QRA Hedge Ratio Computations

ISI Technical Note

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QRA computes hedge ratios for interest rates, exchange rates, and credit risks. Hedge ratios are used in determining replicating portfolios and in assessing value at risk. Hedge ratios are defined by comparing the change in value under small changes in a model factor. QRA computes two different kinds of hedge ratios: differential hedges and Gâteaux hedges.

References: See the Technical Notes, *A Taxonomy of Interest Rate Models and Calibration Techniques* and *Models for Integrating Interest Rate, Sovereign Exchange Rate, and Credit Risk.*

A. Differential Hedge Ratios for Interest Rate Models

Suppose the interest rate is modeled by a single factor spot model. Let $V(r_0)$ denote a security price at time t_0 depending on initial interest rate $r_0 = r(t_0)$. Suppose we perturb this initial interest rate by a small value Δr. The *delta* of V at $r_0 = r(t_0)$ is defined by

$$delta(V; r_0) = \frac{V(r_0 + \Delta r) - V(r_0)}{\Delta r}$$

The gamma is defined by

$$\begin{aligned} gamma(V;\, r_0) &= delta(delta(V;\, r_0);\, r_0) \\ &= \frac{delta(V; r_0 + \Delta r) - delta(V; r_0)}{\Delta r} \\ &= \frac{\dfrac{V(r_0 + \Delta r + \Delta r) - V(r_0 + \Delta r)}{\Delta r} - \dfrac{V(r_0 + \Delta r) - V(r_0)}{\Delta r}}{\Delta r} \\ &= \frac{V(r_0 + 2\Delta r) - 2V(r_0 + \Delta r) + V(r_0)}{\Delta r} \end{aligned}$$

We call these hedges *differental hedges*, since the delta and gamma approximate the first and second derivatives of the Taylor series expansion of V with respect to initial interest rate $r_0 = r(t_0)$.

For multi-factor spot models, $\boldsymbol{r}_0 = \boldsymbol{r}(t_0)$ and delta and gamma are defined as partial derivatives for each interest rate component factor:

$$delta_j(V; \boldsymbol{r}_0) = \frac{V(\boldsymbol{r}_0 + h\boldsymbol{u}_j) - V(\boldsymbol{r}_0)}{h}$$

$$gamma_{jk}\,(V;\, \boldsymbol{r}_0) = delta_{\,j}\,(delta_{\,k}(V;\, \boldsymbol{r}_0);\, \boldsymbol{r}_0)$$

Here, $\boldsymbol{u}_j$ is a unit vector (the zero vector with 1 in the j^{th} coordinate). QRA computes all deltas and non-diagonal gammas.

Differential hedge ratios for forward models can be computed in a similar. Suppose we have n-forward rates packaged as *n*-vector

$$\boldsymbol{f}_0 = < f_1(0,T_1), \ldots, f_n(0,T_n)>$$

where, without loss of generality, we set $t_0 = 0$. Let $V(\boldsymbol{f}_0)$ denote a security price at time t_0 depending on the initial set of forward interest rates. QRA defines the differential hedge delta for forward rate models as a parallel small shift of equal magnitude in all initial forward interest rates:

$$delta(V;\, \boldsymbol{f}_0) = \frac{V(\boldsymbol{f}_0 + h\mathbf{1}) - V(\boldsymbol{f}_0)}{h}$$

$$= \frac{V(f(0,T_1)+h,\, f(0,T_2)+h),\ \ldots,\, f(0,T_N)+h) - V(f(0,T_1),\, f(0,T_2)),\ \ldots,\, f(0,T_N))}{h}$$

Note that **1** is the *N*-vector <1, 1, ..., 1>. In with this definition, there is only one gamma:

$$gamma(V;\, \boldsymbol{f}_0) = delta\,(delta(V;\, \boldsymbol{f}_0);\, \boldsymbol{f}_0)$$

These forward rate hedge ratios are similar to a single factor spot model. Note that to complicate matters, *N* deltas could be defined for each forward rate maturity (yielding N^2 gammas).

B. Gâteaux Hedges

Note that there is no real information incorporated within the hedge ratios associated with the random factors: shifts are all in parallel and of equal magnitude.

There is an alternative way of specifying the small shifts. Here we shift in a magnitude proportional to the diffusion in each factor. If there are *k* factors in a multi-factor model then there are *k* deltas. For example, the forward rate deltas are defined by

$$delta_j(V;\, \boldsymbol{f}_0) = \frac{V(\boldsymbol{f}_0 + h\,\boldsymbol{s}_j(0,\boldsymbol{T};\, \boldsymbol{f}_0)) - V(\boldsymbol{f}_0)}{h}$$

For a k-factor forward model, the k deltas record price changes for yield curve shifts that are "in the direction of" of the standard deviations for the j^{th} factor, j =1..k. The vector $\boldsymbol{s_j}$ corresponds to the j^{th} column in the diffusion matrix $\boldsymbol{S}$ (see equations **F1-F2** in *A Taxonomy of Interest Rate Models and Calibration Techniques*):

$$\boldsymbol{s_j}(0, \boldsymbol{T}; \boldsymbol{f_0}) = < s_{1j}(0, T_1; \boldsymbol{f_0}),\ldots,s_{Nj}(0, T_N; \boldsymbol{f_0}) >$$

Consequently,

$$\begin{aligned} &V(\boldsymbol{f}_0 + h\,\boldsymbol{s}_j(0,\boldsymbol{T};\boldsymbol{f}_0)) = \\ &V(f(0,T_1) + hs_{1j}(0,T_1),\ f(0,T_2) + hs_{2j}(0,T_1),\ ...,\ f(0,T_N) + hs_{Nj}(0,T_N)) \end{aligned}$$

Now there is a gamma for each factor:

$$gamma_{jk}(V; \boldsymbol{f_0}) = delta_j(delta_k(V; \boldsymbol{f_0}); \boldsymbol{f_0})$$

A derivative of a function "in the direction of" another function is actually a Gâteaux derivativ. Consequently, we call this hedging formulation a Gâteaux hedge. These are used in Jarrow (1996).

QRA defines Gâteaux hedges for spot models as well. For k-factor spot models the Gâteaux deltas are:

$$delta_j(V;\boldsymbol{r}_0) = \frac{V(\boldsymbol{r}_0 + hs_j) - V(\boldsymbol{r}_0)}{h}$$

For models where $r = r_1 + r_2 + \ldots + r_n$, $s_j(r_{1,\ldots,} r_n) = S_{jm}$ — the standard deviation of the j^{th} interest rate factor since S_{jm} is a diagonal matrix (see *Taxonomy of Interest Rate Models and Calibration Techniques*, page 2):

$$\boldsymbol{S} = \mathrm{diag}(s_1, s_2, \ldots, s_k)$$

For the k-factor QES models, s_j is also the standard deviation of the j^{th} factor: since S_{jm} is not diagonal, we compute s_j from(see equation **S2**)

$$s^2_j = s^2_{j1} + \ldots + s^2_{jk}$$

the variance of the j^{th} factor.

C. Vega

QRA also computes vega – a measure of the sensitivity to the standard deviation parameters of the interest rate models.

Let $V(\boldsymbol{S}_0)$ denote a security price at time t_0 that depends on the covariance matrix $\boldsymbol{S}$ at time t_0. The differential hedge for *vega* is defined as

$$vega(V;\boldsymbol{S}_0) = \frac{V(\boldsymbol{s_0} + h\mathbf{1}) - \mathbf{V}(\boldsymbol{s_0})}{h}$$
$$= \frac{V(s_1 + h, s_2 + h, \ ..., s_k + h) - V(s_1, s_2, \ ..., s_k)}{h}$$

Note that $\mathbf{1}$ is the N-vector <1, 1, ..., 1>. For spot models where $\boldsymbol{S}$ is a diagonal matrix, the vector $\boldsymbol{s}_0 = \text{diag}(s_1, s_2, \ldots, s_k)$. For k-factor QES models and for k-factor forward rate models, the vector components for $\boldsymbol{s}_0$ are:

$$s_j = s_{j1} + s_{j2} + \ldots + s_{jk}.$$

For Gâteaux hedge,

$$vega(V;\boldsymbol{S}_0) = \frac{V(\boldsymbol{s_0}(\mathbf{1} + h)) - \mathbf{V}(\boldsymbol{s_0})}{h}$$
$$= \frac{V(s_1(1+h), s_2(1+h), \ ..., s_k(1+h)) - V(s_1, s_2, \ ..., s_k)}{h}$$

Note that for forward rate models, the Heath, Jarrow, Morton constraint implies that the drift terms depend on the standard deviation terms. Consequently, changes to the standard deviations change the drifts. In computing the vega hedge for these models, QRA modifies the drift terms.

D. Hedging Ratios for Exchange Rates and Credit

For exchange rates, QRA computes delta, gamma, and vega ratios. QRA models exchange rates as lognormal processes with constant volatility *s*, with drift depending on the difference between the risk free numeraire interest rate and the risk free sovereign interest rate r_n- r_g (*see Models for Integrating Interest Rate, Sovereign Exchange Rate, and Credit Risk*).

Let $V(r_n, r_g, X, s)$ denote the value of a sovereign risky security price at time t_0 depending on the numeraire interest rate r_n, a sovereign rate r_g, today's exchange rate *X*, and the constant standard deviation *s*. (Note that interest rate hedge ratios are computed for both r_n and r_f). For the exchange rate, the differential hedge ratios are

$$delta(V; r_n, r_g, X, s) = \frac{V(r_n, r_g, X+h, s) - V(r_n, r_g, X, s)}{h}$$

$$gamma(V; r_n, r_g, X, s) = delta(delta(V; r_n, r_g, X, s); r_n, r_g, X, s))$$

$$vega(V; r_n, r_g, X, s) = \frac{V(r_n, r_g, X, s+h) - V(r_n, r_g, X, s)}{h}$$

and the Gâteaux hedges are

$$delta(V; r_n, r_g, X, s) = \frac{V(r_n, r_g, X+hs, s) - V(r_n, r_g, X, s)}{h}$$

$$vega(V; r_n, r_g, X, s) = \frac{V(r_n, r_g, X, s(1+h)) - V(r_n, r_g, X, s)}{h}$$

For credit risk, QRA computes a differential hedge delta for an ***increase*** in default probability and a differential hedge delta for a ***decreases*** in recovery rate for securities with non-zero default probabilities and non-zero recovery rates.

Let *V(D, R)* denote the value of a credit risky security at time t_0 depending on default probability *D* and recovery rate R at time t_0. Note that future values of *D* and *R* depend on the evolution of the Markov credit process starting at time t_0.

For the default probability, the differential hedge ratio is

$$delta(V; D, R) = \frac{V(D+\Delta D, R) - V(D, R)}{\Delta D}$$

For the recovery rate, the differential hedge is

$$delta(V; D, R) = \frac{V(D, R+\Delta) - V(D, R)}{\Delta R}$$

E. Replicating Portfolios

One of the uses of hedge ratios is to replicate the behavior of one security V by a *replicating portfolio* of p other securities $V_1,..V_p$, where

$$V = w_1 V_1 + w_2 V_2 + ... + w_p V_p$$

What should the weights be?

If the portfolio is *price neutral* (also called self-financing), then the value of the replicating portfolio is the same as the value of the security. Then we have the relationship:

$$Price(V) = w_1 Price(V_1) + w_2 Price(V_2) + ... + w_p Price(V_p)$$

If the portfolio is *delta neutral,* then the delta of the replicating portfolio is the same as the delta of the security. For example, if the interest rate is modeled by a single factor spot model:

$$delta(V;r_0) = w_1 delta(V_1;r_0) + w_2 delta(V_2;r_0) + ... + w_p delta(V_p;r_0)$$

If the replicating portfolio has p-securities, then we need to find p-relationships in order to use linear algebra to solve a system of p-equations in p-unknowns.

Additional relationships can be found if we require the portfolio be gamma or vega neutral; or if we require the portfolio be credit delta neutral or recovery rate delta neutral.

For k-factor models there are in general k-deltas and more gammas (depending on whether we use Gâteaux and differential hedge.

Note that having these relationships does not guarantee that a replicating portfolio exists. Depending on the specific values, the problem of solving a system of linear equations in p-equations in p-unknowns may have a unique solution, no solution, or a several solutions.

QRA has an interactive utility that builds and solves a system of relationships in order to derive the correct weights of a replicating portfolio. It builds a system of p-equations in p-unknowns from the securities in the replicating portfolio and specific hedge ratios required to be neutralized. It finds the portfolio weights if a solution exists to the system of linear equations.