Technical Note: QRA Value at Risk Computations

ISI Technical Note
Inductive Solutions, Inc.

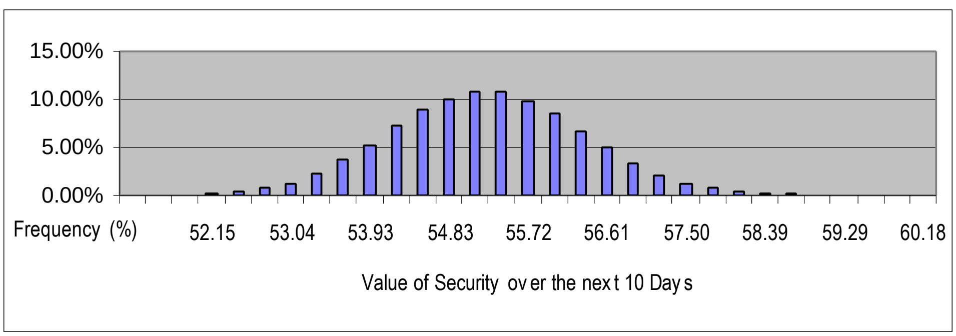
References: See the Technical Notes, *A Taxonomy of Interest Rate Models and Calibration Techniques* and *QRA Hedge Ratio Computations*.

Suppose the current value of a security is V. It may happen, at a time T_h in the future, that the value of the security decreases catastrophically. If the probability of a catastrophic loss is 1% and its respective catastrophic value at time T_h is $V(T_h; 1\%)$ then the *Value at Risk* is defined as

$$VaR = V - V(T_h; 1\%) = \Delta V(T_h; 1\%)$$

Given any time horizon T_h in the future, and given a histogram of possible future values of V, then we can see what the low probability values of V are — we just read them off the histogram.

For example, the following histogram shows the potential values of a security over the next 10 days, together with the probability of the values. It looks like a bell-curve, symmetric around its mean value (approximately 55). Most values cluster around the mean; some are higher and some are lower. A value that represents a loss will be on the left side of the mean value. The histogram shows that the value of the security will be between 53.93 and 56.61 approximately 80% of the time.



The chart also shows that the value of the security will be less 53.93 approximately 10% of the time in the next 10 days. So if the price today was actually 55, then there is a 10% chance that, in the next 10 days, we will lose 55-53.93 = 1.07 — or almost 2% (1.07/55) — of the original value of the security.

For any specified time horizon, QRA simulates the future values of a security and arranges these values in a histogram: given a probability, QRA will compute its value and the resultant VaR.

If $V = V_1 + ... + V_p$ is a portfolio of securities, the portfolio VaR is similarly computed. Given any time horizon T_h in the future, and given a histogram of possible future values of the portfolio, the low probability values of the portfolio are can also be read off the histogram.

Another way of computing VaR is concerned with explicitly looking at the underlying factors that the security (or portfolio) depends on (like interest rate, exchange rate, and credit). We may want to know, for example, "What is the value of V in 5 days, subject to a catastrophic move in interest rates?" (i.e., an event with 1% probability). Or, "What is the value of V in 1 week, subject to a catastrophic move in the exchange rate or credit default probability?" These assessments are done with hedge ratios.

Suppose $V=V(\boldsymbol{r})$ is the price a security depending on a set of n factors, denoted by n-vector $\boldsymbol{r}$ (for example, r can denote a set of component interest rates). Suppose that $\boldsymbol{r}$ evolves according to the rule

$$d\boldsymbol{r} = \boldsymbol{m}dt + \boldsymbol{S}\mathbf{dz}$$

Note that this means that for small changes in time, $\Delta\boldsymbol{r}$ is approximately multivariate normally distributed with mean $m\Delta t$ and covariance $\boldsymbol{SS}^T\sqrt{\Delta t}$. A Taylor series expansion of V implies

$$V(\boldsymbol{r}+\Delta\boldsymbol{r}) = V(\boldsymbol{r}) + V'(\boldsymbol{r})\Delta\boldsymbol{r} + \tfrac{1}{2}\Delta\boldsymbol{r}\, V''(\boldsymbol{r})\Delta\boldsymbol{r} + ...$$

By definition,

$$V'(\boldsymbol{r}) = <\partial V/\partial r_1, \partial V/\partial r_2, ..., \partial V/\partial r_n>$$
$$V''(\boldsymbol{r}) = \{\partial^2 V/\partial r_{jk}\}$$

So

$$delta_j\,(V) = \partial V/\partial r_j$$
$$gamma_{jk}\,(V) = \partial^2 V/\partial r_{jk}$$

If $\Delta\boldsymbol{r}$ denotes a catastrophic interest rate movement at time T_h (for example, a weekly movement with a probability of 1%), then we can approximate the value at time T_h of $V(\boldsymbol{r} + \Delta\boldsymbol{r})$ and derive the VaR for the time horizon and probability —as long as we know the deltas and gammas. This is the second way that QRA computes VaR. Note that the delta and gamma of a portfolio is the sum of the deltas and gammas of the individual portfolio components (since the delta and gamma are *linear* operations).

Note that by Ito's Lemma, security V evolves according to the rule

$$dV = m_V \Delta t + \mathbf{s}_V{}^{\bullet} d\boldsymbol{Z}$$

where

$m_V = \partial V / \partial t \ + \boldsymbol{m} \bullet \partial V / \partial \mathbf{r} \ + \ 1/2 \, \mathrm{trace}[(\boldsymbol{SS}^{\mathrm{T}}) V_{rr}$

$\boldsymbol{s}_V = \partial V / \partial \mathbf{r} \bullet \boldsymbol{S}$

This implies that a small change in *V* is approximately normally distributed with mean $m_V \Delta t$ and variance $\mathbf{s}_V \bullet \mathbf{s}_V \sqrt{\Delta t}$. However, because of correlation effects, the variance of *V* is not necessarily the variance of the sum of the underlying factors. Moreover, if *V* is a portfolio, the variance of *V* is not necessarily the sum of the variance of the underlying component securities.

Other methods that compute VaR (for example, *RiskMetrics*) assume that all security and portfolio returns are zero-mean normally distributed, and that security variance and correlation effects between securities in a portfolio defines market risk. This assumption breaks down in practice, especially because correlations are so dynamic.

The QRA assumption is that security returns are not zero-mean normally distributed and that the risks associated with interest rates, exchange rates, and credit changes outweigh those due to correlation effects. For example, fixed income instruments have "Pull to Par" and "Roll Down" effects. These effects are most observable when computing VaR at time horizons approaching a maturity of a cash flow instrument. This effect was observed to cause incorrect estimates in *RiskMetrics* (see Finger 1996).

QRA accounts for these effects in its computation of VaR using both the histogram method and the hedge ratio method. When using the hedge ratios in computing VaR, QRA assumes that the correlation across different factors (e.g., when combining the standard deviation of exchange rates, currencies, credit, and interest rate) are zero. (Note that this is not the same as assuming that a particular interest rate model has uncorrelated component interest rates or uncorrelated forward interest rates. QRA assumes zero correlation only for "inter-model" combination.)